Cut elimination for provability logic. An unmechanized proof[a]

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Outline

1. *GL* logic and *GLS* system

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Theorem (Löb)

Let $PR(x)$ be a proof predicate for PA and let $\ulcorner x \urcorner$ be the numeral of the number of Gödel of x. For any formula A of PA, if $\vdash_{PA} PR(\ulcorner A \urcorner) \to A$ then $\vdash_{PA} A$.

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$$\frac{A, A, X \vdash Y}{A, X \vdash Y}\ (Ctr) \qquad \frac{X \vdash Y, A, A}{X \vdash Y, A}\ (Ctr)$$

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$$\dfrac{\dfrac{\dfrac{}{\Box(\Box A \to A), \Box A \vdash A, \Box A}\ (IdA) \quad \dfrac{}{A, \Box(\Box A \to A), \Box A \vdash A}\ (IdA)}{\Box(\Box A \to A), \Box A \to A, \Box A \vdash A}\ (\to L)}{\dfrac{\Box(\Box A \to A) \vdash \Box A}{\vdash \Box(\Box A \to A) \to \Box A}\ (\to R)}\ (GLR)$$

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- To achieve this, it was necessary to review the proof scripts and interact with the CAP to follow the reasoning done.
- Unfortunately, due to retrocompatibility issues between the main **Coq** version and the one used by the authors, we were not able to interact with the CAP.
- Due to this, we opted to construct a new proof based on the proof sketches presented by the authors and taking additional guidance from the proof scripts' comments.

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- Unfortunately, two years later Valentini pointed out Leivant's proof was incorrect and presented a new one.
- Valentini's proof uses triple induction over the *grade*, *range* and *width* of a derivation.
- The width of a derivation is defined through (GLR) and (Cut) applications, so it is difficult to handle.

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- Due to these discussions, many authors have tried to get simpler proofs.
- However, most of the alternative proofs have not been completely accepted.
- This lack of acceptance is due to the fact that those proofs are indirect, use semantic elements or use nonstandard sequent calculus.

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- A *regressant* is an expression of the form $X \multimap Y$, associated to a sequent $X \vdash Y$.
- A *regression tree* is essentially a proof search tree build from bottom to top.
- It also has the particularity that in the rule

$$\frac{X, \Box X, \Box A_i \multimap A_i}{W, \Box X \multimap \Box A_1, \ldots, \Box A_n, Z} \; GLR$$

all the possible diagonal formulas are analyzed in different regress trees.

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- The proof proceeds by induction over the formula A and over the height of the highest regress tree for $X \mathrel{\varepsilon\!\!-} Y$.
- It is important to notice that this proof is easier than the older proofs.

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- They also claim that the "regressants detour" is not necessary, so they presented a new proof.
- This proof was done with the **Coq** proof assistant.
- And with the auxiliary system *PSGLS*, which has a terminating proof search and does the regressants' work.

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   These restrictions are introduced to allow us to consider the maximum height derivation for any sequent s ($mhd(s)$), by avoiding infinite proof search.

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Theorem GLS_cut_adm : forall A X_0 X_1 Y_0 Y_1,
(derrec GLS_rules (fun _ => False) (X_0++X_1 , Y_0++A :: Y_1)) ->
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The additive cut rule

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- Possible hardware or software errors.
- Vacuous truths caused by omissions or errors in the implementation.

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Our work

We proved the following theorem and lemmas that were enunciated by Goré et. al. without proof (appealing to the CAP).

Theorem

A sequent $X \vdash Y$ is provable in PSGLS if and only if it is provable in GLS.

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Lemma (Admissibility and invertibility)

The rules (IdA), (Wk), (Ctr) are admissible and the rules $(\rightarrow L)$, $(\rightarrow R)$ are invertible.

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- We also included all the possible cases for r_2 which were omitted (appealing to the CAP) by them.

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- We also included all the possible cases for r_2 which were omitted (appealing to the CAP) by them.
- Finally we concluded that cut is admissible in *GLS* as a corollary.

Our work

- Their proof works directly over *GLS* and invokes *PSGLS* instances of sequent rules when needed.
- We proved the admissibility of cut for *PSGLS* instead.
- Due to this we considered the cases where r_1 and r_2 are instances of (IdB), which are not necessary in their proof.
- We also included all the possible cases for r_2 which were omitted (appealing to the CAP) by them.
- Finally we concluded that cut is admissible in *GLS* as a corollary.
- We could conclude this because we already proved that (Cut) is admissible in *PSGLS* and the proof for the equivalence between *GLS* and *PSGLS* did not use the (Cut) rule.

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- Our proof shows the creativity behind the CAP.
- Our proof shows the innovative techniques used in the CAP in a human readable way.
- This shows that, in this case, the CAP can be trusted from the traditional point of view.

Thank you!