A new proof of Adequacy theorem for a 3-valued logic without deduction-theorem: propositional and quantified cases

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Outline

Overview of Łukasiewicz Logic

- **Infinite-valued Łukasiewicz Logic:** A key non-classical logic, foundational for further developments.
- **MV-Algebras:** Introduced by C. Chang for completeness with Łukasiewicz calculus.
- **Łukasiewicz Implication Algebras:** Algebraic counterparts of implicational fragments in Super-Łukasiewicz logic.
- $\Delta-$**Operator:** Introduced in 1990 by A. V. Figallo to extend 3-valued Super-Łukasiewicz logic.

Research Focus

- **Soundness and Completeness Theorem:** Presentation of a strong version for $\mathrm{C}Ł_3^{\rightarrow,\Delta}$ with respect to its algebraic class.
- **No Deduction Theorem:** $\mathrm{C}Ł_3^{\rightarrow,\Delta}$ does not support the Deduction Theorem, affecting Kalmar's method.
- **Proof Techniques:** Uses a specific homomorphism and Lindenbaum-Tarski method; adaptation for first-order version.
- **Novel Approach:** Differentiates from previous methods in fuzzy logic and related techniques.

Context: Adequacy theorem and its relevance

- The Adequacy theorem ensures that a formula A is provable ($\vdash A$) if and only if it is valid in every model ($\models A$).
- In Łukasiewicz 3-valued logic, this theorem aligns syntactic proofs with semantic truths, making it central to the logic's application.
- Our goal: Prove the Adequacy theorem without relying on the Deduction theorem, which is not applicable in all non-classical logics.

Preliminaries

In this section, we provide the necessary background for setting up our paper. We outline the algebraic properties of the class of 3-valued Lukasiewicz residuation algebras extended by the Δ operator.

Definition ($Ł_3^{\rightarrow,\Delta}$-algebra)

An $Ł_3^{\rightarrow,\Delta}$-algebra is an algebra $(A, \rightarrow, \Delta, 1)$ of type $(2, 1, 0)$ such that $(A, \rightarrow, 1)$ is an $L \rightarrow_3$-algebra and the following identities are satisfied:

$$\Delta x \rightarrow y = x \rightarrow (x \rightarrow y)$$

$$\Delta(\Delta x \rightarrow y) = \Delta x \rightarrow \Delta y$$

We introduce a new binary connective $\Rightarrow$ defined as follows: $x \Rightarrow y := \Delta x \rightarrow y$.

Key Lemmas

Lemma

There exists a lattice-isomorphism between $Con(A)$ and $D(A)$.

Definition (A. Monteiro)

Let A be an $Ł_3^{\rightarrow,\Delta}$-algebra, $D \in D(A)$, and $p \in A$. We say that D is an implicative filter tied to p if $p \notin D$ and for any $D' \in D(A)$ such that $D \subsetneq D'$, then $p \in D'$.

Lemma

Let A be an $Ł_3^{\rightarrow,\Delta}$-algebra and M a maximal implicative filter of A. Then, for every $x \in A \setminus M$, we have that $x \Rightarrow y \in A$ for every $y \in A$.

Main Theorem

Theorem

The variety of $Ł_3^{\to,\Delta}$-algebras is semisimple. Furthermore, the generating algebras are C_3 and the unique subalgebra with support $\{0,1\}$, where the support of C_3 is the set $\{0,\frac{1}{2},1\}$ and the operations $\to$ and Δ are defined by Table.

$\to$	0	$\frac{1}{2}$	1	Δx
0	1	1	1	0
$\frac{1}{2}$	$\frac{1}{2}$	1	1	0
0	0	$\frac{1}{2}$	1	1

Table: The operation of the $Ł_3^{\to,\Delta}$-algebra C_3.

Calculus for $Ł_3^{\to,\Delta}$-algebras

We introduce the Hilbert calculus $CŁ_3^{\to,\Delta}$ for $Ł_3^{\to,\Delta}$-algebras. This calculus is based on the following propositional signature:

- Propositional variables from a denumerable set Var.
- Connectives: implication $\to$ and the operator Δ.

The language of formulas (For) is generated from Var using these connectives.

Axiom Schemes of $CŁ_3^{\to,\Delta}$

Axioms

The calculus $CŁ_3^{\to,\Delta}$ is defined by the following axiom schemes:

1. $\alpha \to (\beta \to \alpha)$
2. $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$
3. $(\alpha \to \beta) \to \beta = (\beta \to \alpha) \to \alpha$
4. $((\alpha \to \beta) \to (\beta \to \alpha)) \to (\beta \to \alpha)$
5. $1 \to \alpha = \alpha$
6. $((\alpha \to (\alpha \to \beta)) \to \alpha) \to \alpha$
7. $\Delta\alpha \to \beta = \alpha \to (\alpha \to \beta)$
8. $\Delta(\Delta\alpha \to \beta) = \Delta\alpha \to \Delta\beta$

Inference Rule: Modus Ponens (MP):

$$\frac{\alpha \quad \alpha \to \beta}{\beta}$$

Adequacy Theorem

Key Propositions:

- Definition of the congruence relation $\equiv$ on For: $\alpha \equiv \beta$ if and only if $\vdash \alpha \to \beta$ and $\vdash \beta \to \alpha$.
- The quotient algebra For$/\equiv$ forms an$Ł_3^{\to,\Delta}$-algebra.

Theorem (Adequacy Theorem)

The calculus $CŁ_3^{\to,\Delta}$ is sound and complete with respect to the class of $Ł_3^{\to,\Delta}$-algebras.

- **Soundness:** *If* $\vdash \alpha$, *then* $\models \alpha$.
- **Completeness:** *If* $\models \alpha$, *then* $\vdash \alpha$.

This establishes the strong adequacy of the calculus $CŁ_3^{\to,\Delta}$.

Introduction to $\forall Ł_3^{\rightarrow,\Delta}$

The first-order logic of $CŁ_3^{\rightarrow,\Delta}$ extends the propositional system $CŁ_3^{\rightarrow,\Delta}$. The language now includes two quantified symbols $\forall$ and $\exists$, along with the set of individual variables V.
A structure $\mathcal{S} = \langle S, P, F, C, \cdot \rangle$ for a signature Σ is defined where:

- S is the domain,
- P are predicate symbols,
- F are function symbols, and
- C are individual constants.

Axiom Schemas for $\forall Ł_3^{\rightarrow,\Delta}$

Let Σ be the first-order signature. The logic $\forall Ł_3^{\rightarrow,\Delta}$ over Σ is defined by extending the logic $CŁ_3^{\rightarrow,\Delta}$ expressed in the new language by adding the following:

- **($\forall$1)** $\varphi(x/t) \rightarrow \exists x\varphi$, if t is free for x in φ.
- **($\forall$2)** $\forall x\varphi \rightarrow \varphi(x/t)$, if t is free for x in φ.
- **($\forall$3)** $\Delta\exists x\varphi \leftrightarrow \exists x\Delta\varphi$.
- **($\forall$4)** $\Delta\forall x\varphi \leftrightarrow \forall x\Delta\varphi$.

Inference rules such as $(\forall R1)$ and $(\forall R2)$ extend the system by allowing quantified terms in deductions.

Main Theorem for $\forall Ł_3^{\to,\Delta}$

Given a structure $\mathcal{S}$ and a valuation v, the truth value of a formula is defined as:

- $||P(t_1, \ldots, t_n)||_v = P^{\mathcal{S}}(||t_1||_v, \ldots, ||t_n||_v)$.
- $||\varphi \to \psi||_v = ||\varphi||_v \to ||\psi||_v$.
- $||\Delta\varphi||_v = \Delta||\varphi||_v$.
- $||\forall x\varphi||_v = \bigwedge_{a\in S} ||\varphi[x \to a]||_v$.
- $||\exists x\varphi||_v = \bigvee_{a\in S} ||\varphi[x \to a]||_v$.

Theorem (Soundness and Completeness): The first-order logic $\forall Ł_3^{\to,\Delta}$ is sound and complete with respect to $Ł_3^{\to,\Delta}$-algebras.

- **Soundness:** If $\vdash \varphi$, then $\models \varphi$.
- **Completeness:** If $\models \varphi$, then $\vdash \varphi$.

Conclusions

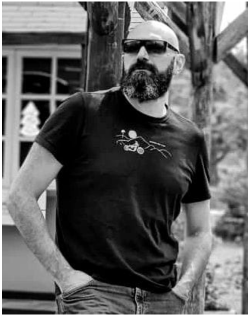
In the book **"Paraconsistent Logics and Logics of Formal Inconsistency"** by Carnielli and Coniglio, the authors presented Adequacy Theorems for several paraconsistent logics at the propositional level. In Chapter 4, they built homomorphisms for three-valued logics using Blok-Pigozzi's method.

We followed a similar approach in this paper, using a homomorphism for $CŁ_3^{\rightarrow,\Delta}$, which was first introduced in **"Algebraic Models of Lukasiewicz Logic"** by Esteva and Godo. Our contribution includes a new syntactic proof for the **Homomorphism Construction Theorem**.

Key Result: We extended this approach to first-order logic, proving Adequacy Theorems without relying on algebraic properties of the first-order Lindenbaum-Tarski algebra. This offers a more "syntactic" methodology, as demonstrated in **"Lindenbaum-Tarski Algebras for Fuzzy Logics"** by Rasiowa.

This syntactic method can also be applied to other three-valued logics studied in **"Algebraizable Logics"** by Blok and Pigozzi. It simplifies existing algebraic approaches while maintaining rigorous logical consistency.

Tribute to Aldo Figallo (In Memoriam)



- Aldo Figallo, a dear friend and collaborator, made significant contributions to this work.
- His dedication and passion for logic were unmatched.
- We honor his memory and continue his legacy through this work.
- Aldo passed away in June 2024, but his influence and contributions remain with us.